Preview of Sets for Mathematics

Contructions and theorems to be discussed

Before beginning the study of foundational matters, a student should have considerable experience with the mathematics that motivates this study. We begin by mentioning some of that mathematics with an emphasis on some of the phenomena we want to understand.

After beginning with counting one learns operations like addition, multiplication or greatest common divisor on domains of numbers. These are examples of *mappings*, since to any pair of numbers they associate a unique resulting number. In fact, counting itself is a mapping: to each whole number the counting operation assigns a next number. We begin our study with an analysis of mappings and their domains (sets). We will immediately see (Section 1.1) that the most important feature of mappings is that they can be *composed*. The various uses of mappings for analysis of their codomains (listings, parameterizations) and of their domains (properties) will be exemplified with some simple examples from everyday experience (1.2). We will also see that mappings and their composition provide the data for a *category*—the structure which axiomatizes the important properties of composition (1.4). By noting the existence of a special set 1, we can define *elements* of a set as mappings with domain 1 and also observe that 1 serves to distinguish between mappings (1.5).

Forming new sets by joining distinct sets together is a basic operation. We study that operation and its definition by mappings (2.1) and then consider parts of sets as special mappings. The important point here is that the property of belonging to a part can be represented by another mapping whose codomain is a fixed set of truth values (2.4).

When some experience with mappings is gained, one considers equations such as $x^2 + 1 = 5$. Here the left hand side is a mapping which associates to each number x in a specified domain, for example the real numbers $\mathbb{R}$, a unique value (here the result of squaring and adding 1). The right hand side is to be viewed as the *constant* mapping on the same domain whose unique value is 5 for any input (or domain) value. *Solving* the equation is determining the part of the domain where the two mappings described are equal, and we will study such parts, or *equalizers* in Section 3.4. More operations for combining sets and diagrams of sets are seen to be defined by mapping properties (3.3, 3.5).

Partitioning a set is a process for specifying in which part each element of the set lies, i. e. a mapping which is *surjective* (4.4). We will see that to require, for every partition, the possibility of a choice of representatives for it is a powerful condition known as the axiom of choice (4.3, Appendix B). In fact the axiom nearly characterizes the constant sets as opposed to the more cohesive or more variable sets.

The subject of calculus courses is the properties (especially of continuity, smoothness and integrability) of mappings from (parts of) spaces to spaces. The mappings encountered in these courses are between parts of Euclidean spaces, for example intervals in the real line $\mathbb{R}$ or domains in the plane or higher dimensional space. Smoothness of such mappings is measured by differentiability, but differentiation is an operator which associates the derivative function to a differentiable function. Similarly, the integral is an operator on continuous, or more generally integrable, functions. In Chapter 5 we will see how *mapping sets* are the basis for understanding multivariable mappings and *functionals* like the derivative and integral. Not only mappings of a single time or space parameter such as $f(x) = x^2 + 1 : \mathbb{R} \to \mathbb{R}$, but also differentiation and integration properties of mappings of several parameters like $f(x,t) = x^2 + t/x - t : [0,1] \times [0,\infty)\{x \neq t\} \to \mathbb{R}$ are important in calculus. Solutions to differential equations are (parameterized) mappings which make equal the operators defining the left and right side of the equation. Initial conditions are points which specify particular mappings.

Recall that matrices provide a representation (once bases are chosen) for the *linear* mappings (or *transformations*) between vector spaces. Spaces of (linear) functionals are important domains to be parameterized too, though we will not study this example directly. The linearity of the vector spaces makes their behaviour, in some ways, quite different from the behaviour of sets.

Based on the ideas of category, part, constructing objects from universal mapping properties, and mapping sets, we can specify what is meant by a category of abstract sets and arbitrary mappings. In Chapter 6 we pause to summarize and observe some of the relationships among the axioms we have been studying. Then we look at a first example of variable sets, wherein the variation is parameterized by two stages (the current and a previous stage). In this first example of variable sets the 'sets' are actually modeled by mappings of constant sets and the 'mappings' between them involve equations among composed mappings. The three global truth values of this category demonstrate that even very simple variation refutes the logical law of the excluded middle.

Exponentiation of sets has some properties that are expressed in the term 'duality' which is met in linear algebra when considering linear transformations to the one-dimensional space. The properties we find in Chapter 7 are similar and prepare for consideration of the distributive law as it applies to sets and for Cantor's famous *diagonal argument.*

A mapping from a set to itself (a self- or 'endo'-mapping) provides a process for specifying a next (or 'successor') element from any given one. Sets equipped with just such a simple dynamic provide another example of the notion of variable set. Moreover, the selection of a starting element combined with an endo-mapping specifies a *sequence* of elements. A sequence where each of the elements is different matches the intuitive idea of a set which can have no finite number and so is *infinite.* A 'natural numbers object' is a set equipped with a starting element ('zero') and a self-mapping ('successor') which can serve as the domain of all sequences recursively definable in its category. Such an object is needed for some purposes in our category of sets and mappings. If it is present, all of arithmetic is available inside the category.

More domains of variation are easily found; they often take the form of various sorts of graphs. The study of several examples of such domains and of the nature of the sets varying over them is made possible by our knowledge of abstract sets and arbitrary mappings. The beginning of such a study is the goal of the final chapters of this book.

Reference to Appendix A

In Appendix A on 'Logic as the algebra of parts', the role of basic logical operators in elementary algebra is explained. Since that role is the same in this book and in all mathematical subjects, that appendix should be read several times.

The relationship to computer science

To understand both data structures and algorithms requires the use of the idea of category because the algorithms implement mappings which have definite domains and codomains. For example, the Euclidean algorithm returns the greatest common divisor of a pair of input integers.

Any algorithm to sort an array takes an array as input and provides a unique (sorted!) array as output. In this case the domain data structure, the array, has several operations (= mappings) available. For example, take any array and position in the array and return the value of the array element at the position.

In another example, a stack S and a stack element x uniquely determine the stack S' which results from pushing x onto S. Often algorithms output *truth values*, for example an array search algorithm returns *'true'* exactly when the input is found in the array.

Also in computer science the study of functional programming languages such as Lisp, Scheme and ML has benefited from categorical foundations. One example is the study of *polymorphism* or the construction of algorithms which are available for inputs of various types. Understanding the role of mapping sets (or exponentials) is essential to the so-called λ-abstraction operation.

Presenting data structures with operations and equations has provided a clear foundation, and has also made it clear exactly what are the operations necessary to a complete presentation of particular data structures. For example the stack data structure can be presented with exactly 2 operations and 2 equations (see Walters book, *Categories and Computer Science*), something that may not be clear from many Computer Science texts. More recently studying the construction of computing devices from input/output components has led to a new comprehension of 'feedback' and its role.

A note on the exercises

To develop an understanding of the material in this book it is vitally important to complete the exercises of which there are two types: The exercises that are contained in the text should be done in the course of reading. They complete and extend the material as it is being presented, and it is essential that these be understood before proceeding. Most of them have quite short solutions and will become obvious with the drawing of a simple diagram. At the end of each Chapter is a section of Additional Exercises. Some of these provide additional examples of the concepts in the chapter and tests of understanding of the text material. Some provide a preview of later material; others develop ideas which are not included in the main text and give additional examples of concepts from mathematics to which the main ideas apply.