Near Distributive Laws

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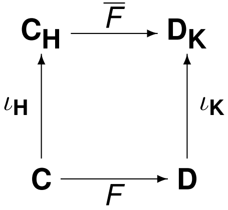
Preliminaries

$\mathbf{H} = (H, \mu, \eta)$ a monad in **C**, $\mathbf{K} = (K, \nu, \rho)$ a monad in **D**

$\mathbf{C_H}$ for the Kleisli category of **H**

$\mathbf{D^K}$ for the Eilenberg-Moore category of **K** algebras

A **Kleisli lifting** of functor $F : \mathbf{C} \to \mathbf{D}$ is $\overline{F} : \ \mathbf{C_H} \to \mathbf{D_K}$

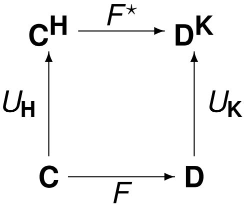


Kleisli liftings are classified exactly by natural transformations $\lambda : FH \to KF$ satisfying certain axioms.

$\overline{F}(f : A \to HB) = \lambda \circ (F\ f)$

Introduction

Likewise An **Eilenberg-Moore lifting** of functor $F : \mathbf{C} \to \mathbf{D}$ is $F^\star : \mathbf{C}^{\mathbf{H}} \to \mathbf{D}^{\mathbf{K}}$



E-M liftings are classified exactly by natural transformations $\sigma : KF \to FH$ again satisfying certain axioms.

λ and σ are denoted **lifting transformation** and their axioms guarantee that $\overline{F}$ and $F^\star$ are functorial.

Introduction

Well known question: How to compose two monads?

The composition of (H, μ, η) and $K, \nu, \rho)$ should have form $(KH, \tau, \rho\eta)$.

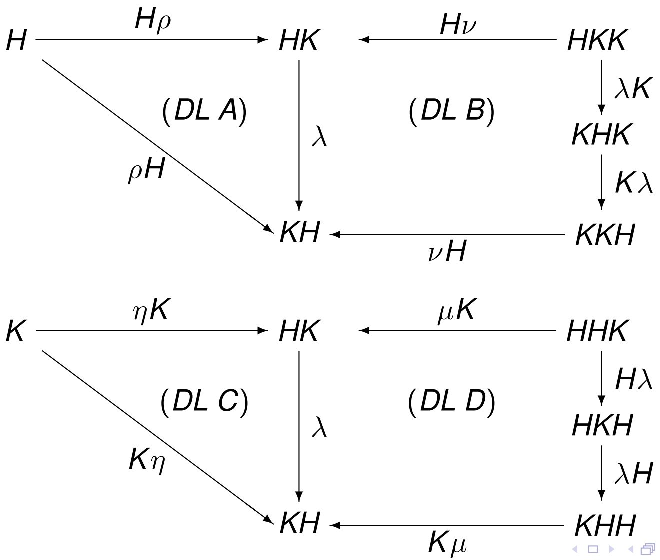
The problem is that there is no obvious τ.

The solution is to provide a natural transformation $\lambda : HK \to KH$ which allows τ to be defined as $\tau = KHKH \xrightarrow{K\lambda H} KKHH \xrightarrow{\mu\nu} KH$

The axioms on λ which enable $(KH, \tau, \rho\eta)$ to be a monad(i.e. that λ is a **distributive law**) were first discovered by Beck.

Introduction

$\lambda : HK \to KH$ a **distributive law of H over K** if



Introduction

Prior slide: The two diagrams are the axioms for the lifting transformations $\lambda : HK \to KH$ (where λ is both a Kleisli and an E-M lifting transformation for $\overline{H}$ and $K^{\star}$ respectively) of the monads H and K.

Difficulties in composing monads generally arise from the axioms as opposed to the naturality requirement.

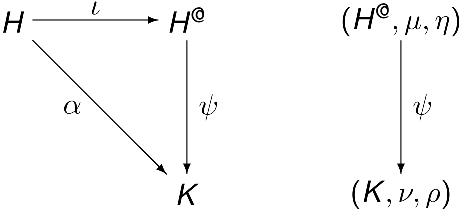
While composition of monads can be achieved for many monads in programming, it often comes at a cost in terms of the definition of λ

One approach is to only require (DL C, DL D) to hold in which case we say λ is a **near distributive law**.

We present two approaches to building near distributive laws via **free monads** and **pre-monads**.

Free Monads

Our most relaxed model of a monad is a functor $H : \mathcal{V} \to \mathcal{V}$. If it exists, the **free monad generated by** H is $(H^{@}, \mu, \eta; \iota)$ where $\mathbf{H}^{@} = (H^{@}, \mu, \eta)$ is a monad in $\mathcal{V}$ and $\iota : H \to H^{@}$ is a natural transformation, subject to the universal property



that if (K, ν, ρ) is a monad in $\mathcal{V}$ and $\alpha : H \to K$ is a natural transformation then there exists a unique monad map ψ as shown with $\psi\iota = \alpha$.

Free Monads

Example

Assume that $\mathcal{V}$ has finite powers. For a finite ordinal $i \geq 1$, let $H_i : \mathcal{V} \to \mathcal{V}$ be the functor $H_i X = X^i$, the usual i-product functor. When $\mathcal{V} = \mathbf{Set}$, the data type $H_i^{@} X$ is the set of all i-ary trees in which every node is either an element of X, denoted $L_i x$ (a leaf) or has i subtrees beneath it, denoted $B_i t_1 \cdots t_i \in H_i^{@} X$. The natural transformation $\eta_X : X \to H_i^{@} X$ maps x to $L_i x$ while $\mu_X : H_i^{@} H_i^{@} X \to H_i^{@}$ maps $L_i t$ to t and $B_i\ tt_1 \ldots\ tt_i$ to $B_i\ (\mu_X tt_1)\ \ldots (\mu_X tt_i)$.

For now we consider only H for which $\mathbf{H}^{@}$ exists.

Theorem

$\mathcal{V}^H$ is isomorphic over $\mathcal{V}$ to the category of Eilenberg-Moore algebras $\mathcal{V}^{\mathbf{H}^{@}}$. The isomorphism $\Phi : \mathcal{V}^{\mathbf{H}^{@}} \to \mathcal{V}^H$ is given by

$$\Phi(X,\, H^{@} X \xrightarrow{\xi} X) \;=\; (X,\, HX \xrightarrow{\iota_X} H^{@} X \xrightarrow{\xi} X)$$

Free Monads

Definition

Let $H : \mathcal{V} \to \mathcal{V}$ generate a free monad $\mathbf{H}^{@}$ and let $K : \mathcal{V} \to \mathcal{V}$ be a functor. Let $K^{\star} : \mathcal{V}^{\mathbf{H}^{@}} \to \mathcal{V}^{\mathbf{H}^{@}}$ be a functorial lift of K with lifting natural transformation $\lambda^{@} : H^{@}K \to KH^{@}$. We say $K^{\star}$ is a **flat** functorial lift if there exists a natural transformation $\lambda : HK \to KH$ such that the following square commutes.

$$\begin{array}{ccc} HK & \xrightarrow{\iota K} & H^{@}K \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda^{@}} \\ KH & \xrightarrow[K\iota]{} & KH^{@} \end{array} \qquad (3)$$

We then say that λ **generates** $K^{\star}$, or λ **generates** $\lambda^{@}$, and when K is a monad that $\lambda^{@}$ is a **flat near-distributive law**.

Free Monads

Theorem

Given $H, K : \mathcal{V} \to \mathcal{V}$ such that $\mathbf{H}^{@}$ exists, every natural transformation $\lambda : HK \to KH$ generates a flat functorial lift of K through $\mathbf{H}^{@}$.

Proof.

Given λ, define $K^{\dagger} : \mathcal{V}^{H} \to \mathcal{V}^{H}$ over $\mathcal{V}$ by

$$K^{\dagger}(X, \delta) \;=\; (KX,\; HKX \xrightarrow{\lambda_X} KHX \xrightarrow{K\delta} KX)$$

If $f : (X, \delta) \to (Y, \epsilon)$ is an H-homomorphism, the diagram

$$\begin{array}{ccccc}
HKX & \xrightarrow{\lambda_X} & KHX & \xrightarrow{K\delta} & KX \\
{\scriptstyle HKf}\downarrow & & {\scriptstyle KHf}\downarrow & & \downarrow{\scriptstyle Kf} \\
HKY & \xrightarrow[\lambda_Y]{} & KHY & \xrightarrow[K\epsilon]{} & KY
\end{array}$$

Free Monads

shows that $Kf : K^{\dagger}(X,\delta) \to K^{\dagger}(Y,\epsilon)$ is again an H-homomorphism.
We then have the functorial lift

$$K^{\star} \;=\; \mathcal{V}^{\mathbf{H}^{@}} \xrightarrow{\;\Phi\;} \mathcal{V}^{H} \xrightarrow{\;K^{\dagger}\;} \mathcal{V}^{H} \xrightarrow{\;\Phi^{-1}\;} \mathcal{V}^{\mathbf{H}^{@}}$$

□

Corollary

Given $H, K : \mathcal{V} \to \mathcal{V}$ where $\mathbf{K}$ is a monad and $\mathbf{H}^{@}$ exists, then every natural transformation $\lambda : HK \to KH$ generates a flat near distributive law $\lambda^{@} : H^{@}K \to KH^{@}$.

Near Distributive Laws for Free Monads

Definition

For functor $F : \mathcal{V} \to \mathcal{V}$, a ***pre-strength*** *on F is a pair (F, Γ^F) where Γ^F is a natural transformation*
$\Gamma^F_{V_1 \cdots V_n} : FV_1 \times \cdots \times FV_n \to F(V_1 \times \cdots \times V_n)$
A morphism $\alpha : (F, \Gamma^F) \to (G, \Gamma^G)$ is a natural transformation $\alpha : F \to G$ such that the following square commutes.

$$\begin{array}{ccc}
FV_1 \times \cdots \times FV_n & \xrightarrow{\alpha_{V_1} \times \cdots \times \alpha_{V_n}} & GV_1 \times \cdots \times GV_n \\
\Big\downarrow \Gamma^F_{V_1 \cdots V_n} & & \Big\downarrow \Gamma^G_{V_1 \cdots V_n} \\
F(V_1 \times \cdots \times V_n) & \xrightarrow[\alpha_{V_1 \times \cdots \times V_n}]{} & G(V_1 \times \cdots \times V_n)
\end{array}$$

This forms a category of prestrengths of order n

Near Distributive Laws for Free Monads

Lemma

For any monad $\mathbf{K} = (K, \nu, \rho)$ *in* **Set** *there exists a generic prestrength* $\Gamma_n : KA_1 \times ...KA_n \to K(A_1 \times ...A_n)$ *of dimension* $n \geq 1$.

$\Gamma_1 = \mathrm{id}_X$

Γ_2 formed using the Kock prestrength construction.

For $m_1, m_2 \in KX$ define
$\Gamma_2(m_1, m2) = m1 >>= \lambda a \to (m2 >>= \lambda b \to \rho(a, b))$ where
for $f : X \to KY$, $(m >>= f) = f^{\#} m$

Straightforward application of monad laws shows
$K(f \times g) \circ \Gamma_2(m_1, m_2) = \Gamma_2 \circ (Kf \times Kg)(m_1, m_2)$ and so Γ_2 is natural.

Near Distributive Laws for Free Monads

Proceeding inductively, if $\Gamma_i : (KX)^i \to KX^i$ is natural, we obtain a natural transformation

$$KX \times (KX)^i \xrightarrow{id_X \times \Gamma_i} KX \times KX^i \xrightarrow{\Gamma_{XX^i}} K(X^{i+1})$$

Lemma

$\Gamma_2 : KA \times KB \to K(A \times B)$ is also associative, namely $\Gamma_2 \circ (\Gamma_2 \times 1)(m1, m2, m3) = \Gamma \circ (1 \times \Gamma)(m1, m2, m3)$.

Amenable Monads

Difficult to find nontrivial monads which admit a distributive law with every monad.

Definition

A monad $\mathbf{H}$ in $\mathcal{V}$ is **amenable** if for every monad $\mathbf{K}$ in $\mathcal{V}$, K has a functorial lift through $\mathcal{V}^{\mathbf{H}}$.

Proposition

The monads $\mathbf{H}_{\mathbf{i}}^{©}$ in **Set** *of Example 1 are amenable.*

Proof.

Let $\mathbf{K} = (K, \nu, \rho)$ be a monad in **Set**. By previous lemma there exists a generic natural transformation $\Gamma_i : H_i K \to K H_i$ for every $i \geq 1$. By letting $\lambda = \Gamma_2 = \Gamma_{xx}$ in flat near-distributive result, we are done. □

Amenable Monads

Theorem

The list monad is amenable.

For any semigroup $(X, \cdot)$, define the binary operation on $K^*(X, \cdot)$ as $(KX, \cdot\cdot)$ where $k_1 \cdot \cdot k_2 = (K\cdot) \circ \Gamma_2(k_1, k_2)$. If $f : (X, \cdot) \to (Y, \cdot)$ is a semigroup morphism then so is $K^* f$ since

$$
\begin{aligned}
& (Kf)(k_1 \cdot \cdot_X k_2) \\
= \;& (Kf) \circ (K\cdot_X) \circ \Gamma_{KX,KX}(k_1, k_2) \\
= \;& (K\cdot_Y) \circ K(f \times f) \circ \Gamma_{KX,KX}(k_1, k_2) \qquad (f \text{ a semigp hom}) \\
= \;& (K\cdot_Y) \circ \Gamma_{KY,KY} \circ (Kf \times Kf)(k_1, k_2) \qquad (\Gamma \text{ natural}) \\
= \;& (Kfk_1) \cdot \cdot_Y (Kfk_2)
\end{aligned}
$$

The result easily extends to empty lists by defining $K^*(X, \cdot, e_X)$ as $(KX, \cdot\cdot, \rho(e_X))$.

Amenable Monads

Example

We apply $K^*(X, \cdot) == (KX, \cdot\cdot)$. *When* K *is:*

the exception monad $KX = X + 1$, *then* $(X + 1, \cdot\cdot)$ *is the obvious semigroup defined by* $a_1 \cdot \cdot a_2 = a_1 \cdot a_2$ *when* a_1, a_2 *are in* X, $*$ *otherwise.*

the reader monad $KX = C \times X$ *for commutative monoid* $(C, *)$, $(c_1, x_1) \cdot \cdot (c_2, x_2) = (c_1 * c_2, x_1 \cdot x_2)$.

the writer monad $KX = A \to X$, *and* t_i *in* KX, $t_1 \cdot \cdot t_2 = \lambda x \to t_1 x \cdot \ t_2 x$.

the state monad $KX = S \to X \times S$, *and* t_i *in* KX, $t_1 \cdot\cdot\ t_2 = \lambda s \to$ *let* $(x_1, s_1) = t_1 s$ *in let* $(x_2, s_2) = t_2 s_1$ *in* $(x_1 \cdot x_2, s_2)$.

Amenable Monads

Example

For **K** the reader monad $KX = C \times X$, $\lambda = \Gamma_2 : H_2K \to KH_2$ becomes $\Gamma_2((c_1, x_1), (c_2, x_2)) = (c_1 * c_2, (x_1, x_2))$. Acting on a binary tree t of type $H_2^@ KX$, $\lambda^@(t) = (p, t^*)$ where p is the product of the c_i's found in the leaves and t^* is the corresponding tree in $H_2^@ X$ consisting only of the elements of X.

Example

When K is $H_j^@$, we can give a recursive construction of the functorial lift of lifting K through $\mathbf{Set}^{H_i^@}$ defining the near-distributive law $\lambda : H_i^@ H_j^@ \to H_j^@ H_i^@$ in cases. For $i, j \geq 1$:

$$
\begin{aligned}
\lambda(L_i L_j a) &= L_j L_i a \\
\lambda L_i(B_j t_1 \cdots t_j) &= B_j(\lambda L_i t_1) \cdots (\lambda L_i t_j) \\
\lambda B_i(tt_1 \cdots tt_i) &= (H_j^@ B_i)\Gamma_i(\lambda tt_1) \cdots (\lambda tt_i)
\end{aligned}
$$

where tt_i *has type* $H_i^@ H_j^@$

Amenable Monads

Proposition

Let $\mathcal{V}$ have small coproducts, let (H_α) be a small family of endofuctors and let $H = \coprod H_\alpha$ be the pointwise coproduct. Assume that the free monads $\mathbf{H}_\alpha^@$, $\mathbf{H}^@$ exist. Then if each $\mathbf{H}_\alpha^@$ is amenable, so is $\mathbf{H}^@$.

Example

Let Σ be a disjoint sequence (Σ_n) of (possibly empty) sets. A Σ-algebra is (X, δ) where X is a set and $\delta = (\delta_\sigma : \sigma \in \Sigma)$ with $\delta_\sigma : X^n \to X$ if $\sigma \in \Sigma_n$. Defining

$$H_\Sigma X = \coprod_{\sigma \in \Sigma_n} X^n$$

then H_Σ-algebra is the same thing as a Σ-algebra. $H_\Sigma^@ X$ is the usual free Σ-algebra generated by X and is an an amenable monad in **Set**.

Prestrengths and Flat Near-Distributive Laws

Lemma

For any monad $\mathbf{K} = (K, \nu, \rho)$ *in* **Set**, *if there exists a natural transformation* $\gamma : K \to id$ *then there exists a prestrength* $\Gamma_i : KA_1 \times ... \times KA_i \to K(A_1 \times ... \times A_i)$ *for any* $i \geq 1$.

Proof.

The construction is simple: for $i = 1$ define $\Gamma_1 = \rho \circ \gamma$. If $i \geq 2$ then $\Gamma_i = \rho \circ (\gamma \times \cdots \times \gamma)$. Since in each case Γ_i is a composition of natural transformations we are done. □

Proposition

For any monad $\mathbf{K} = (K, \nu, \rho)$ *with any* $\gamma : K \to id$ *as in the previous lemma, there exists a flat near-distributive law* $\lambda^{@} : \mathbf{H_i^{@}} K \to K\mathbf{H_i^{@}}$.

Proof.

For any $i \geq 1$, the prestrength Γ_i of the previous lemma generates a natural transformation $H_i K \to K H_i$ and so the result follows immediately from Corollary 1.1. □

Prestrengths and Flat Near-Distributive Laws

Example

For $j \geq 1$ let γ denote the j-th projection natural transformation $\Pi_j : H_j \to id$. By the previous proposition this generates a flat near-distributive law $\lambda^{@} : \mathbf{H_i^{@}} H_j^{@} \to H_j^{@} \mathbf{H_i^{@}}$ which generally differs from the earlier examples.

Example

For monad $\mathbf{K}$ the M-**Set** monad $KA = C \times A$ for C a commutative monad with identity e, $\gamma : K \to id$ defined as $\gamma(c, a) = a$ is clearly natural thus generating $\Gamma_n : KA_1 \times ...KA_n \to K(A_1 \times ...A_n)$ by $\Gamma_n((c_1, a_1), ...(c_n, a_n)) = \rho(a_1, ...a_n) = (e, (a_1, ...a_n))$. The resulting flat distributive law $\lambda^{@} : \mathbf{L}(C \times A) \to C \times \mathbf{L}A$ takes $[(c_1, a_1), ...(c_n, a_n)]$ to $(e, [a_1, ...a_n])$.

Uniformly branching trees and non-flat near-distributive laws

Motivation: There exists a full distributive law $\lambda : LL \to LL$. It exploits the observation: algebras on L are semigroups. The corresponding distributive law does not arise via a flat lifting.

Question: Can we generalize this to free monads? In the case of near-distributive laws, yes.

Recall that an algebra for $H_i^{@}$ is generated by $(A, [\,]_i)$, where $[\,]_i : A^i \to A$ is an i-ary operation on A. For $i, j \geq 1$, we build a recursive schema for canonical functorial liftings of $H_j^{@}$ over $\mathbf{Set}^{H_i^{@}}$. To do this, we define $(H_j^{@})^*$ in cases and expressly define $(H_j^{@})^*(A, [\,]_i) = (H_j^{@} A, [\,]_i)$. (Note that we use the same notation for the two i-ary operations). When $i = 1$

- $[(L_j\ a)]_1 = L_j([\ a]_1)$
- $[(B_j\ t_1 ... t_j)]_1 = B_j\ [t_1]_1 ... [t_j]_1$

Uniformly branching trees and non-flat near-distributive laws

Likewise when $j = 1$ we have

- $[L_1 a_1, ... L_1 a_i]_i = L_1[a_1, ... a_i]_i$
- $[L_1 a_1, ... L_1 a_{i-1}, (B_1\ t)]_i = B_1\ [L_1 a_1, ... L_1 a_{i-1}, t]_i$
- *etc*
- $[(B_1\ t_1)\ t_2 ... t_i]_i = B_1\ [t_1,\ t_2 ... t_i]_i$

Otherwise for $i, j \geq 2$

- $[L_j a_1, ... L_j a_i]_i = L_j[a_1, ... a_i]_i$
- $[L_j a_1, ... L_j a_{i-1}, (B_j\ t_{i,1} ... t_{i,j})]_i = B_j\ [L_j a_1, ... L_j a_{i-1}, t_{i,1}]_i\ t_{i,2} ... t_{i,j}$
- *etc*
- $[(B_j\ t_{1,1} ... t_{1,j})\ t_2 ... t_i]_i = B_j\ t_{1,1} ... t_{1,j-1}\ [t_{1,j},\ t_2 ... t_i]_i$

Uniformly branching trees and non-flat near-distributive laws

A near-distributive law λ is created via the lifting functor $(H_j^{@})^*$ over $H_i^{@}$ algebras described above.

Applying $(H_j^{@})^*$ to $(H_i^{@}A, B_i)$, the i-ary operation associated to the canonical algebra $(H_i^{@}A, \mu)$ generates λ defined by the following set of equations:

- $\lambda(L_i L_j\ a) = L_j L_i\ a$
- $\lambda L_i(B_j t_1 ... \ t_j) = B_j(\lambda\ L_i t_1)...(\lambda\ L_i t_j)$
- $\lambda(B_i\ tt_1 ... \ tt_i) = [\lambda tt_i]_i$ where $[\ ]_i$ was defined previously

Uniformly branching trees and non-flat near-distributive laws

Example

When $i = 1$ $H_1^{@}$ is the M-set or writer monad $N \times \_$ where N is the commutative monoid of natural numbers $\{0, 1, 2, \dots\}$ under addition and $\lambda : N \times H_j^{@}\ a \to H_j^{@}(N \times A)$ is actually a distributive law.

Likewise when $j = 1$, $\lambda : H_i^{@}(N \times A) \to N \times H_i^{@}A$ can be described by: for an arbitrary tree tt in $H_i^{@}(N \times A)$, $\lambda\ tt = (k, t^*)$ where t^* is the tree in $H_i^{@}A$, with the same shape as tt, generated by replacing every leaf in tt of the form $L_i(m, a)$ by $L_i a$ and where k equals the sum of all the various m's found in the leaves. Again λ is a full distributive law.

Uniformly branching trees and non-flat near-distributive laws

Proposition

For any $i, j \geq 2$ the near distributive law $\lambda : H_i^{@} H_j^{@} \to H_j^{@} H_i^{@}$ above fails to produce a full distributive law as one can produce a generic tree $t \in H_i^{@} H_j^{@} H_j^{@}$ for which law (DLB) fails.

Proof.

For $\lambda : H_i^{@} H_j^{@} \to H_j^{@} H_i^{@}$ we produce $t \in H_i^{@} H_j^{@} H_j^{@}$ with $4(j-1) + i$ leaves for which (DL B) fails. Let

- $lt = B_j\ (L_j\ (L_j\ a_1))...(L_j\ (L_j\ a_{j-1}))\ (L_j(B_j(L_j a_j)...(L_j a_{2j-1})))$
- $rt = B_j\ (L_j(B_j(L_j a_{2j})...(L_j a_{3j-1})))\ (L_j\ (L_j\ a_{3j}))...(L_j\ (L_j\ a_{4j-2}))$
- $t = B_i(L_i(lt))\ (L_i(L_j(L_j b_1)))\ ...\ (L_i(L_j(L_j b_{i-2})))\ (L_i(rt))$

□

Pre-Monads

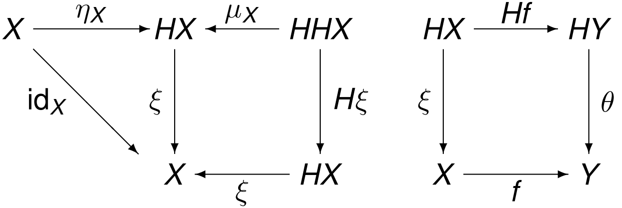
Definition

A **pre-monad** in $\mathcal{V}$ is $\mathbf{H} = (H, \mu, \eta)$ with $H : \mathcal{V} \to \mathcal{V}$ a functor and with $\eta : \mathrm{id} \to H$, $\mu : HH \to H$ natural transformations.

Composition of pre-monads: For pre-monads (H, μ, η) and (K, ν, ρ), natural transformation $\lambda : HK \to KH$ generates the **composite** pre-monad

$$(KH,\ KHKH \xrightarrow{K\lambda H} KKHH \xrightarrow{\nu\mu} KH,\ \mathrm{id} \xrightarrow{\rho\eta} KH)$$

The axioms defining an **algebra** (X, ξ) for a pre-monad $\mathbf{H} = (H, \mu, \eta)$ and an $\mathbf{H}$-homomorphism $f : (X, \xi) \to (Y, \theta)$ are exactly the same as for a monad, namely



Pre-Monads

Proposition

Let (L, m, e) be the list monad in **Set***. Modify this to the pre-monad $(L, m, \hat{e})$ where $\hat{e}_X\, x = [x, x]$. Then* $\mathbf{Set}^{(L,m,\hat{e})}$ *is the category of bands (semigroups in which every element is idempotent).*

Proposition

Pre-monads may be equivalently described as $(H, (\cdot)^{\#}, \eta)$ where $H : \mathcal{V} \to \mathcal{V}$ is a functor, $\eta : id \to H$ is a natural transformation and $X \xrightarrow{f} HY \mapsto HX \xrightarrow{f^{\#}} HY$ is an operator subject to the axioms

$$\begin{array}{ll} \text{(PME.1)} & g : Y \to HZ, \quad g^{\#} = HY \xrightarrow{Hg} HHZ \xrightarrow{(\mathrm{id}_{HZ})^{\#}} HZ \\ \text{(PME.2)} & \text{For } f : X \to HY,\, g : Y \to Z, \;\; (Hg)\, f^{\#} = ((Hg)f)^{\#} \end{array}$$

$$f^{\#} = HX \xrightarrow{Hf} HHY \xrightarrow{\mu_Y} HY \tag{1}$$

$$\mu_X = (\mathrm{id}_{HX})^{\#} \tag{2}$$

Pre-Monads

Definition

A **pre-monad map** $\sigma : (H, \mu, \eta) \to (K, \nu, \rho)$ is a natural transformation $\sigma : H \to K$ such that $\sigma \circ \eta = \rho$ and $\nu \circ \sigma\sigma = \sigma \circ \mu$ (same as for monads), so monads form a full subcategory of pre-monads.

Definition

Given a pre-monad $\mathbf{H}$ in $\mathcal{V}$, a **monad approximation** of $\mathbf{H}$ is a reflection $\sigma : \mathbf{H} \to \mathbf{K}$ of $\mathbf{H}$ in the full subcategory of monads.

Theorem

Let $\mathbf{H} = (H, \mu, \eta)$, $\mathbf{K} = (K, \nu, \rho)$ be pre-monads in $\mathcal{V}$. Then a pre-monad map $\sigma : H \to K$ induces a functor $W : \mathcal{V}^{\mathbf{K}} \to \mathcal{V}^{\mathbf{H}}$ over $\mathcal{V}$ defined by

$$W(X, \xi) \quad = \quad (X,\ HX \xrightarrow{\sigma X} KX \xrightarrow{\xi} X) \tag{3}$$

Pre-Monads

If, additionally, $\mathbf{K}$ is a monad, then $\sigma \mapsto W$ is bijective with inverse

$$\sigma_X = HX \xrightarrow{H\rho_X} HKX \xrightarrow{\gamma_X} KX \tag{4}$$

where $(KX, \gamma_X) = W(KX, \nu_X)$.

Theorem

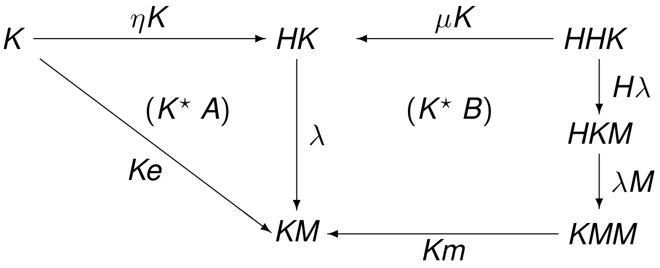
Let $\mathbf{H}$ *be a pre-monad such that* $U : \mathcal{V}^{\mathbf{H}} \to \mathcal{V}$ *is monadic so that there exists a monad* $\mathbf{K}$ *and an isomorphism of categories* $\Phi : \mathcal{V}^{\mathbf{K}} \to \mathcal{V}^{\mathbf{H}}$ *over* $\mathcal{V}$*. Then the corresponding pre-monad map* $\sigma : \mathbf{H} \to \mathbf{K}$ *of the previous theorem is a monad approximation of* $\mathbf{H}$*.*

Near Distributive Laws for Pre-Monads

The laws DL A, DL B, DL C, DL D make sense whenever (H, μ, η), (K, ν, ρ) are pre-monads. The following generalizes the idea of E-M liftings and near distributive laws.

Theorem

Let $K : \mathcal{V} \to \mathcal{V}$ be a functor, (M, m, e) a monad in $\mathcal{V}$ and let (H, μ, η) be a pre-monad in $\mathcal{V}$ such that $\mathcal{V}^{\mathbf{H}} \to \mathcal{V}$ is monadic. Functorial lifts $K^\star : \mathcal{V}^{\mathbf{M}} \to \mathcal{V}^{\mathbf{H}}$ correspond bijectively to natural transformations $\lambda : HK \to KM$ which satisfy $(K^\star A, K^\star B)$:



Near Distributive Laws for Pre-Monads

The correspondences are

$$K^\star(X, MX \xrightarrow{\theta} X) \quad = \quad (KX, HKX \xrightarrow{\lambda_X} KMX \xrightarrow{K\theta} KX) \quad (5)$$

and, if $K^\star(MX, m_X) = (KMX, \gamma_X)$,

$$\lambda_X \quad = \quad HKX \xrightarrow{HKe_X} HKMX \xrightarrow{\gamma_X} KMX \qquad (6)$$

Moreover, half of this result holds if **M** is only a pre-monad, namely if λ satisfies ($K^\star$ A) and ($K^\star$ B), then $K^\star$ as in (5) is a functorial lift $\mathcal{V}^{\mathbf{M}} \to \mathcal{V}^{\mathbf{H}}$ of K.

We would like to connect this to our prior example of the bands monad. The following result does the trick.

Near Distributive Laws for Pre-Monads

Theorem

Let $\mathbf{K} = (K, \nu, \rho)$ *be a pre-monad in* $\mathcal{V}$ *and let* $\mathbf{H} = (H, \mu, \eta)$ *be a pre-monad in* $\mathcal{V}$ *with monad approximation* $\sigma : \mathbf{H} \to \widehat{\mathbf{H}}$*,* $\widehat{\mathbf{H}} = (\widehat{H}, \widehat{\mu}, \widehat{\eta})$*. Let* $\lambda : HK \to KH$ *be a natural transformation satisfing (DL C, DL D). Then there exists a near distributive law* $\widehat{\lambda} : \widehat{H}K \to K\widehat{H}$ *of* $\widehat{\mathbf{H}}$ *over* $\mathbf{K}$ *such that the following square commutes. We say* λ ***generates*** $\widehat{\lambda}$*.*

$$\begin{array}{ccc} HK & \xrightarrow{\sigma K} & \widehat{H}K \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\widehat{\lambda}} \\ KH & \xrightarrow[K\sigma]{} & K\widehat{H} \end{array} \qquad (33)$$

Near Distributive Laws for Pre-Monads

Lemma

Let $\mathbf{H} = (H, \mu, \eta)$ be a pre-monad in $\mathcal{V}$ with monad approximation $\sigma : \mathbf{H} \to \widehat{\mathbf{H}} = (\widehat{H}, \widehat{\mu}, \widehat{\eta})$. Let $s : \widehat{H} \to H$ be a section of σ, that is, s is a pre-monad map with $\sigma s = 1$. Then $\widehat{\mathbf{H}}$ satisfies $\widehat{\eta} = \sigma \circ \eta$ and for map $f : X \to \widehat{HY}$, $\widehat{f^{\#}} : \widehat{HX} \to \widehat{HY} = \sigma \circ (s_Y \circ f)^{\#} \circ s_X$.

Example

Let (L, m, e) be the list premonad where $e(x) = [x, x]$ and $m\ ll = [fst(fst\ ll), lst(lst\ ll)]$. The reflection $\sigma\ [x] = (x, x)$ and $\sigma\ [x_1, \ldots x_n] = (x_1, x_n)$ defines the monad approximation of $\sigma : (L, m, e) \to (B, \mu, \eta)$ where $\mathbf{B}$ is the rectangular band monad $B\ A = A \times A$. σ has an obvious section $s(x, y) = [x, y]$ and so we can derive $\mu(a, b, c, d) = \sigma_X \circ m \circ L(s_X) \circ s_X(a, b, c, d)$ $= \sigma_X \circ m \circ L(s_X)[(a, b), (c, d)] = \sigma_X \circ m[[a, b], [c, d]] =$ $\sigma_X[a, d] = (a, d)$ as expected.

Near Distributive Laws for Pre-Monads

When $\mathcal{V} = \textbf{Set}$, the image of $\sigma : H \to \widehat{H}$ is a submonad with the universal property. Thus all monad approximations are pointwise split epic in **Set**.

Theorem

If $\sigma : \mathbf{H} \to \widehat{\mathbf{H}}$ is a pointwise split epic monad approximation then if $\lambda : HK \to KH$ is a distributive law then so too is $\widehat{\lambda} : \widehat{H}K \to K\widehat{H}$.

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